ENDOTRIVIAL MODULES FOR FINITE GROUP SCHEMES II

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ABSTRACT. It is well known that if G is a finite group then the group of endotrivial modules is finitely generated. In this paper we prove that for an arbitrary finite group scheme G, and for any fixed integer $n > 0$, there are only finitely many isomorphism classes of endotrivial modules of dimension n. This provides evidence to support the speculation that the group of endotrivial modules for a finite group scheme is always finitely generated. The result also has some applications to questions about lifting and twisting the structure of endotrivial modules in the case that G is an infinitesimal group scheme associated to an algebraic group.

1. INTRODUCTION

Let S be a finite group scheme over a field k. The endotrivial modules for S form an important class of modules which, among other things, determine self equivalences of the stable category of S-modules, modulo projective S-modules. In the case that G is the scheme of a finite p-group, there is a complete classification of endotrivial modules [7]. This classification has been extended to the group algebras of many other families of finite groups (cf. [2, 4, 5]).

An endotrivial module is an S-module M with the property that $\mathrm{Hom}_k(M, M) \cong M \otimes M^* \cong k \oplus P$ (as S-modules) where P is a projective S-module. Two endotrivial modules M and N are equivalent if there exist projective modules P and Q such that $M \oplus P \cong N \oplus Q$. The equivalence classes of endotrivial S-modules forms an abelian group $T(S)$ under tensor product. It was shown by Puig [11] that this group is finitely generated in the case that S is a finite group. This fact also follows from the classification. For arbitrary finite group schemes, it is an open question as to whether $T(S)$ is finitely generated.

One of the main ingredients in proving that $T(S)$ is finitely generated is a demonstration that for any fixed non-negative integer $n \geq 0$, there are only finitely many endotrivial modules of dimension equal to n. When S is a unipotent group scheme this was proved by the authors in [3, Theorem 3.5]. In Section 2, we extend these earlier results by showing that for arbitrary finite group schemes there are only finitely many endotrivial modules for a given dimension. This result will be referred to as the "Finiteness Theorem".

The Finiteness Theorem has some very strong connections with the notion of lifting endotrivial modules to action of H where H is a group scheme containing S as a normal subgroup scheme. In the case when H is connected, the Finiteness Theorem implies that every endotrivial S-module is H-stable (i.e., the twists of an endotrivial module M^h are

all isomorphic to M (as S-module) for all $h \in H$). Different notions of lifting such as "tensor stability" and "numerical stability" have been investigated recently by Parshall and Scott [10]. In Section 3, we outline these various definitions, and we introduce the new concept of lifting called "stably lifting" which entails lifting S-modules to H-structures in the stable module category for S. We connect our new notion of stable lifting with the ideas presented by Parshall and Scott.

Let G be semisimple algebraic group scheme, B a Borel subgroup with unipotent radical U defined and split over $\mathbb{F}_p$, and let k be an algebraically closed field of characterstic p. Let G_r, B_r, U_r denote their r^{th} infinitesimal Frobenius kernels. The existence of the Steinberg module shows that every projective G_r-module (resp. B_r, U_r) lifts to G (resp. B, U). Furthermore, if M stably lifts then one can use this result to show that all syzygies $\Omega^n(M)$ stably lift. In [3, Theorem 5.7, Theorem 6.1], $T(B_1)$ and $T(U_1)$ was completely determined for all primes. By using this classification, we prove that all B_1 (resp. U_1) endotrivial modules stably lift to B (resp. U). We suspect this will also hold for G_r. Finally, we exhibit an endotrivial module for B_1, namely $\Omega^2(k)$ when the root system is of type A_2 and $p = 2$, which does not admit a B-structure.

2. The finiteness theorem

We begin by introducing the basic definitions which will be used throughout this paper. Let S be a finite group scheme. We will consider the category $\mathrm{Mod}(S)$ of rational S-modules and the stable module category $\mathrm{StMod}(S)$. Since S is a finite group scheme the notion of projectivity is equivalent to injectivity. For M and N objects in $\mathrm{Mod}(S)$, we say that $[M] = [N]$ in $\mathrm{StMod}(S)$ if and only if $M \oplus P \cong N \oplus Q$ for some projective S-modules, P and Q.

Suppose that S is any finite group scheme defined over k. An S-module is an endotrivial module provided that, as S-modules,

$$\mathrm{Hom}_k(M, M) \cong k \oplus P$$

for some projective S-module P. Equivalently, an S-module M is endotrivial module if $[M^* \otimes M] = [k]$ in $\mathrm{StMod}(S)$. Note for any S-module M there exists a canonical isomorphism $\mathrm{Hom}_k(M, M) \cong M^* \otimes M$. We can now define the group $T(S)$ of endotrivial S-modules as follows. The objects in $T(S)$ are the equivalence classes $[M]$ in $\mathrm{StMod}(S)$ of endotrivial S-modules. The group operation is given by $[M] + [N] = [M \otimes N]$. The identity element is the class $[k]$ and the inverse of $[M]$ is the class $[M^*]$. The group $T(S)$ is abelian because the associated coproduct used to construct the action of S on the tensor product of modules is cocommutative.

In this section, we show that the number of endotrivial modules for a finite group scheme H having any particular dimension is finite. The proof follows somewhat the same lines if that in [3] and also [11], and is based on an idea of Dade [8].

Suppose that k is an algebraically closed field. In this section it will be more convenient to work with finite-dimensional cocommutative Hopf algebras. For a finite group scheme H, let $A = k[H]^*$, the group algebra of H. There is an equivalence of categories between

H-modules and A-module. Furthermore, the finite dimensional k-algebra A is a cocommutative Hopf algebra. As a consequence, projective A-modules are also injective. Let $P_1, \ldots, P_r$ be a complete set of representatives of the isomorphism classes of projective indecomposable A-modules. Each P_i has a simple top $S_i = P_i/\operatorname{Rad}(P_i)$ and a simple Socle $T_i = \operatorname{Soc}(P_i)$. The collection $\{S_1, \ldots, S_r\}$ is a complete set of isomorphism classes of simple modules, as is the set $\{T_1, \ldots, T_r\}$. We need not assume here that $T_i \cong S_i$, though this is often the case.

For each $i = 1, \ldots, r$, we assume that P_i is a left ideal in A. That is, we assume that $P_i = Ae_i$ for e_i a primitive idempotent in A. For each i choose a nonzero element $u_i \in T_i \subseteq P_i$. Then u_i has the property that $Au_i = T_i$. Note that since $u_i \in P_i$, we have that $u_i = u_i e_i$ and that $u_i P_i \neq \{0\}$.

Lemma 2.1. *Let i be an integer between 1 and r. Suppose that M is an A-module and that $u_i M \neq \{0\}$. Then M has a direct summand isomorphic to P_i. Moreover, if $t_i = \operatorname{Dim}(u_i P_i)$ is the rank of the operator of left multiplication by u_i on P_i and $a_i = \operatorname{Dim}(u_i M)/t_i$, then*

$$M \cong P_i^{a_i} \oplus M'$$

where M' has no direct summands isomorphic to P_i.

Proof. Let $m \in M$ be an element such that $u_i m \neq 0$. Then, $u_i m = u_i e_i m$. Define $\psi : P_i \longrightarrow M$ by $\psi(ae_i) = ae_i m$ for any a in A. This is well defined since $P_i = Ae_i$. Now, $\psi(u_i) \neq 0$ and hence $\psi(T_i) \neq \{0\}$. Therefore, ψ is injective. Because, A is a self-injective ring, the image of ψ is a direct summand of M and hence $M \cong P_i \oplus N$ for some submodule N of M. This proves the first statement.

The second statement follows by an easy induction beginning with the observation that (as vector spaces)

$$u_i M \cong u_i P_i \oplus u_i N.$$

$\square$

For a positive integer n, let $\mathcal{V}_n$ denote the variety consisting of all representations of the algebra A of dimension n. It is defined as follows. Suppose that the collection $a_1, \ldots, a_t$ is a chosen set of generators of the algebra A, so that every element of A can be written as a polynomial in (noncommuting) variables $a_1, \ldots, a_t$. We fix this set of generators for the remainder of the discussion in this section. Then we have that $A \cong k\langle a_1, \ldots, a_t \rangle / \mathcal{J}$ for some ideal $\mathcal{J}$. A representation of dimension n of A is a homomorphism $\theta : A \longrightarrow M_n(k)$, where $M_n(k)$ is the ring of $n \times n$ matrices over k. The representation is completely determined by the assignment to each a_i of an $n \times n$ matrix $\theta(a_i) = (a_{rs}^i)$.

For the purposes of defining the variety $\mathcal{V}_n$, we consider the polynomial ring $R = k[x_{rs}^i]$ in tn^2 (commuting) variables with $1 \leq i \leq t$, $1 \leq r, s \leq n$, and the assignment

$$a_i \quad \leftrightarrow \quad (x_{rs}^i) \in M_n(R)$$

for $i = 1, \ldots, t$. The ideal $\mathcal{J}$ determines an ideal $\mathcal{I}$ in the ring R. That is, any relation $f(a_1, \ldots, a_t)$ in $\mathcal{J}$, when converted into an expression on the matrices using the above assignment, defines a collection of relations, one for each r and s in the elements of R. For example, if it were the case that $a_1 a_2 = 0$ in A, then the polynomial $a_1 a_2$ would be an element of $\mathcal{J}$, and for each r and s, the polynomial $\sum_{u=1}^n x_{ru}^1 x_{us}^2$ would be an element of $\mathcal{I}$.

Lemma 2.2. *Suppose that M is an A-module and P is an indecomposable projective A-module. Let s be a nonnegative integer. Let $\mathcal{W}$ be the subset of $\mathcal{V}_n$ consisting of all representations σ of A having the property that $M \otimes L_\sigma$ has no submodule isomorphic P^s, where L_σ is the A-module affording σ. Then $\mathcal{W}$ is a closed set in $\mathcal{V}_n$.*

Proof. By the previous lemma, there is an element $u \in A$ such that, if t is the rank of the matrix of u acting on P, then P^s is isomorphic to a submodule of $M \otimes L_\sigma$ if and only if the rank of the matrix $\mathcal{M}_u$ of u on $M \otimes L_\sigma$ is at least st. Now we observe that u is a polynomial in the (noncommuting) generators of A and hence the entries of the matrix of u on L_σ are polynomials in the entries of the matrices of the generators of A on L_σ. If we fix a representation of M, then the entries of the matrix of u on M are elements of the base field k. It follows that the entries of the matrix $\mathcal{M}_u$ of u on $M \otimes L_\sigma$ are all polynomials in the variables of the ring $R = k[x_{rs}^i]$. Likewise, the determinant of any $st \times st$ submatrix of $\mathcal{M}_u$ is a polynomial in the variables of R. As a consequence, the condition that every such determinant is zero (which is the same as saying that the rank of $\mathcal{M}_u$ is less than st) defines a closed set in $\mathcal{V}_n$. $\square$

Lemma 2.3. *Suppose that M is an endotrivial A-module of dimension n. Let $\mathcal{W}$ be the set of all representations σ in $\mathcal{V}_n$ such that the module L_σ afforded by σ is not isomorphic to $M \otimes \chi$ for any one dimensional A-module χ. Then $\mathcal{W}$ is a closed set in $\mathcal{V}_n$.*

Proof. Suppose that N is a one dimensional A-module. Because M is endotrivial, we have that $N \otimes M$ is endotrivial. Thus

$$(N \otimes M) \otimes (N^* \otimes M^*) \cong k \oplus \sum_{i=1}^{r} P_i^{n_i}$$

where $P_1, \ldots, P_r$ are the indecomposable projective A-modules and $n_1, \ldots, n_r$ are nonnegative integers. For each i, let $\mathcal{W}_i$ be the set of all $\sigma \in \mathcal{V}_n$ with the property that $L_\sigma \otimes N^* \otimes M^*$ does not contain a submodule isomorphic to $P_i^{n_i}$. The sets $\mathcal{W}_i$ are closed by Lemma 2.2. Hence, the set $\mathcal{U}_N = \mathcal{W}_1 \cup \cdots \cup \mathcal{W}_r$ is also closed. If σ is not in $\mathcal{U}_N$, then

$$L_\sigma \otimes N^* \otimes M^* \quad \cong \quad U \oplus \sum_{i=1}^{r} P_i^{n_i}$$

for some A-module U. But the dimension of U must be one because $\dim L_\sigma \otimes N^* \otimes M^* = \dim(N \otimes M) \otimes (N^* \otimes M^*)$. Therefore,

$$L_\sigma \otimes N^* \otimes M^* \otimes U^* \quad \cong \quad k \oplus \sum_{i=1}^{r} U^* \otimes P_i^{n_i}$$

and hence, $L_\sigma \cong U \otimes N \otimes M$. Now we claim that $\mathcal{W} = \cap_N \mathcal{U}_N$ where N runs through the one dimensional A-modules. So $\mathcal{W}$ is closed. $\square$

At this point we are ready to prove our main theorem.

Theorem 2.4. *For any natural number n, there is only a finite number of isomorphism classes of endotrivial modules of dimension n.*

Proof. Suppose that M is an indecomposable endotrivial module of dimension n. Let $\mathcal{U}$ be the subset of $\mathcal{V}_n$ consisting of representations σ with the property that the underlying module L_σ is isomorphic to $N \otimes M$ for some A-module N of dimension one. Note that A has only finitely many isomorphism classes of dimension one, and hence there are only finitely many isomorphism classes of modules represented in $\mathcal{U}$.

By Lemma 2.3, $\mathcal{U}$ is an open set in $\mathcal{V}_n$. Hence, $\overline{\mathcal{U}}$, the closure of $\mathcal{U}$ is a union of components in $\mathcal{V}_n$. Therefore, the theorem is proved with the observation that $\mathcal{V}_n$ has only finitely many components. $\square$

3. LIFTINGS AND STABILITY

Let S be a finite group scheme which is a normal subgroup scheme in a group scheme H. In this section we will describe different notions of when an S-module has a structure that extends to H.

We say that an S-module M *lifts* to H if M has an H-module structure whose restriction to S agrees with the (original) S-module structure. This is the strongest form of "lifting". The weakest form of lifting is the notion of H-stable. Let M be a S-module. For $h \in H$, one can consider the twisted module M^h which is a $S' = h^{-1}Sh$-module (cf. [9, I. 2.15]). In particular if h normalizes S then the twisted module M^h is an S-module. An S-module M is called H-*stable* if and only if $M^h \cong M$. If the S-module M lifts to H-module, then M is H-stable. The converse statement is not true as we will see in Section 6 (cf. [10, 4.2.1]).

Following [10, 2.2.2, 2.2.3] we recall the notions of numerical and tensor stablity defined by Parshall and Scott.

Definition 3.1. *An S-module M is* numerically H-stable *if there exists an H-module Z such that $Z|_S \cong M^{\oplus n}$.*

Definition 3.2. *An S-module M is* tensor H-stable *if there exists a finite-dimensional H/S-module Y such that $M \otimes Y$ is an H-module whose restriction to S coincides with $M \otimes (Y|_S)$.*

Tensor H-stability is equivalent to numerically H-stable (cf. [10, 2.2.3]). It is clear that if M lifts to H then M is tensor H-stable and numerically H-stable. Furthermore, tensor H-stable and numerically H-stable imply H-stable.

Next we introduce a new concept of lifting which will be relevant for our study of endotrivial modules.

Definition 3.3. *An S-module M* stably lifts *to H if there exists an H-module K such that $K|_S \cong M \oplus P$ where P is a projective S-module.*

Observe that in the definition of stable lifting to H, the S-modules K and M represent the same object in $\mathrm{StMod}(S)$, the stable category of all S-modules. If M lifts H then M stably lifts to H. Also, if M is non-projective as an S-module and stably lifts to H then by using the Krull-Schmidt theorem and the fact that twists of projective S-modules are projective, it follows that M is H-stable.

4. Applications

Suppose that G is a semisimple, simply connected algebraic group, defined and split over the finite field $\mathbb{F}_p$ with p elements for a prime p. Let k be the algebraic closure of $\mathbb{F}_p$. Let Φ be a root system associated to G with respect to a maximal split torus T. Let Φ^+ (resp. Φ^-) be the set of positive (resp. negative) roots and Δ be a base consisting of simple roots. Let B be a Borel subgroup containing T corresponding to the negative roots and let U denote the unipotent radical of B. More generally, if $J \subset \Delta$, let L_J be the Levi subgroup generated by the root subgroups with roots in Δ, P_J the associated (negative) parabolic subgroup and U_J its unipotent radical such that $P_J = L_J \ltimes U_J$.

Let H be an affine algebraic group scheme over k and let $H_r = \ker F^r$. Here $F : H \to H^{(1)}$ is the Frobenius map and F^r is the rth iteration of the Frobenius map. We note that there is a categorical equivalence between modules for the restricted p-Lie algebra $\text{Lie}(H)$ of H and H_1-modules. For each value of r, the group algebra kH_r is the distribution algebra $\text{Dist}(H_r)$ (cf. [9]). In general, for the rest of this paper, we use $\text{Dist}(H_r)$ to denote the group algebra of H_r.

For any group scheme H, let $\text{mod}(H)$ be the category of finite dimensional rational H-modules. This construction can be applied when $H = G$, B, P_J, L_J, U, U_J and T. Note that the use of T (maximal torus) and $T(-)$ (endotrivial group) will be clear from the context. Let $X := X(T)$ be the integral weight lattice obtained from Φ. The set X has a partial ordering: if $\lambda, \mu \in X$, then $\lambda \geq \mu$ if and only if $\lambda - \mu \in \sum_{\alpha \in \Pi} \mathbb{N}\alpha$.

Let $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$ is the coroot corresponding to $\alpha \in \Phi$. The set of dominant integral weights is defined by

$$X_+ := X(T)_+ = \{\lambda \in X(T) : \ 0 \leq \langle \lambda, \alpha^\vee \rangle \text{ for all } \alpha \in \Delta\}.$$

Furthermore, the set of p^r-restricted weights is

$$X_r(T) = \{\lambda \in X : \ 0 \leq \langle \lambda, \alpha^\vee \rangle < p^r \text{ for all } \alpha \in \Delta\}.$$

Let $X(T_r)$ be the set of characters of T_r which can be identified with the set of one dimensional simple modules for T_r.

For a reductive algebraic group G, the simple modules are labelled $L(\lambda)$ and the induced modules are $H^0(\lambda) = \text{ind}_B^G \lambda$, where $\lambda \in X(T)_+$. The Weyl module $V(\lambda)$ is defined as $V(\lambda) = H^0(-w_0\lambda)^*$. Let $T(\lambda)$ be the indecomposable tilting module with highest weight λ.

We can now apply the Finiteness Theorem to demonstrate, under mild assumptions on H, that every endotrivial module is H-stable. In the following sections we show that the problem of lifting of endotrivial modules is rather subtle.

Theorem 4.1. *Let H be a connected affine algebraic group scheme and S be a finite group scheme which is a normal subgroup scheme of H. If M is an endotrivial S-module then M is H-stable.*

Proof. Consider the closed subgroup $A = \{h \in H : \ M^h \cong M\}$ in H. According to the finiteness theorem, this must have finite index in H because there are only finitely many endotrivial S-modules of any fixed dimension. Therefore, A must contain the connected component of H. Because H is connected, we have that $A = H$, which proves the theorem. $\square$

Corollary 4.2. *Let $H = G$, B, P_J, L_J, U, U_J and T as above, and H_r be the rth Frobenius kernel. If M is an endotrivial H_r-module, then M is H-stable.*

Another application of the Finiteness Theorem involves proving that the restriction of an endotrivial G_r-modules to conjugate unipotent radicals of Borel subgroups produces syzygies of the same degree.

Theorem 4.3. *Let U and U' be the unipotent radicals of the Borel subgroups B and B'. If M is an endotrivial G_r-module with*

$$M|_{U_r} \cong \Omega_{U_r}^{n_1}(k) \oplus (proj)$$

and

$$M|_{U'_r} \cong \Omega_{U'_r}^{n_2}(k) \oplus (proj)$$

then $n_1 = n_2$.

Proof. The Borel subgroups B and B' are conjugate by some $w \in W$, and $B' = B^w = w(B)w^{-1}$. According to Corollary 4.2, we have $M \cong M^w$ as G_r-modules. Under this isomorphism U_r is isomorphic to U'_r and $M|_{U_r}$ can be identified with $M|_{U'_r} = M^w$. The result now follows by applying these isomorphisms. $\qquad\square$

5. Lifting Endotrivial Modules

Let M be an S-module and $\Omega_S^n(M)$ $(n = 0, 1, 2, \dots)$ be the nth syzygy of M obtained by taking a projective resolution of M. We will assume that $\Omega_S^n(M)$ has no projective S-summands. By taking an injective resolution of M we can define $\Omega_S^n(M)$ for n negative. Note that if M is endotrivial over S then $\Omega_S^n(M)$ is an endotrivial module for all $n \in \mathbb{Z}$. The following result provides conditions on when there are stable liftings for the syzygies of an H-module M (when considered as an S-module).

Proposition 5.1. *Let S be a finite group scheme which is a normal subgroup scheme of H. Suppose there exists a projective S-module P which lifts to an H-module. Furthermore, suppose that there exists a surjective H-map $P \to k$. If M is a finite-dimensional H-module then $\Omega_S^n(M)$ stably lifts to H for each $n \in \mathbb{Z}$.*

Proof. Consider the surjective H-module homomorphism $P \to k$. We will prove that $\Omega_S^n(M)$ stably lifts by induction on $n \in \mathbb{N}$. For $n \leq 0$ a similar inductive argument can be used.

For $n = 0$, $\Omega_S^0(M) = M$ which is an H-module so we can set $K_0 = M$. Now assume that $K_n|_S \cong \Omega_S^n(M) \oplus Q_n$ where K_n is an H-module K_n and Q_n is a projective S-module. Now define K_{n+1} as the kernel in the short exact sequence obtained by tensoring the complex $P \to k$ by K_n:

$$0 \to K_{n+1} \to P \otimes K_n \to K_n \to 0.$$

Then K_{n+1} is an H-module with $K_{n+1}|_S \cong \Omega_S^{n+1}(M) \oplus Q_{n+1}$ for some projective S-module Q_{n+1}. $\qquad\square$

Let G be a reductive group with subgroups P, B and U as before. The existence of the Steinberg representation St_r can be used to prove that every projective G_r (resp. P_r, B_r, U_r) module stably lifts to G (resp. P, B, U). It is only known for $p \geq 2(h-1)$ that projective G_r-modules lift to G, but there is strong evidence this holds for all p.

We can now prove that for reductive groups and their associated Lie type subgroups that the syzygies of the trivial module lift stably. The proof also utilizes the existence of the Steinberg representation.

Theorem 5.2. *Let $H = G$ (resp. P, B, U) and $S = G_r$ (resp. P_r, B_r, U_r). For each $n \in \mathbb{Z}$, $\Omega_S^n(k)$ stably lifts to H.*

Proof. It suffices to prove the theorem in the case that $H = G$ and $S = G_r$. The other cases will follow by restriction. Let $\mathrm{St}_r = L((p^r - 1)\rho)$ be the Steinberg module, and set $P := \mathrm{St}_r \otimes L((p^r - 1)\rho)$. Then there exists a surjective G-module homomorphism $P \to k$ [9, II 10.15 Lemma]. The result now follows by Proposition 5.1 ☐

We note that it is not trivial to prove the fact that the left U_r-module structure on $\mathrm{Dist}(U_r)$ lifts to $H = U$. The conjugation action of U_r on $\mathrm{Dist}(U_r)$ lifts to U and there exists a U-module map $\mathrm{Dist}(U_r) \to k$ under the conjugation action. However, the module $\mathrm{Dist}(U_r)$ is not a projective module under this action (i.e., the conjugation action does not lift the left action of $\mathrm{Dist}(U_r)$ on itself).

Corollary 5.3. *Let $H = B$ (resp. U), and $S = B_r$ (resp. U_r). Then every endotrivial S-module lifts stably to an H-module.*

Proof. We first consider the case that $H = B$ and $S = B_r$. According to [3, Theorem 6.1, 6.2], $T(B_r) \cong X(T_r) \times T(U_r)$. The one dimensional B_r endotrivial modules corresponding elements of $X(T_r)$ are all B-modules. Therefore, it suffices to prove the statement when $H = U$ and $S = U_r$.

Assume that Φ is not A_2 in the case that $p = 2$ and $r = 1$. Then any endotrivial B_r-module is isomorphic to $\Omega_{B_r}^n(\lambda)$ for some $\lambda \in X_r(T)$. Since λ lifts to a B-module, by Theorem 5.2, $\Omega_{B_r}^n(\lambda)$ stably lifts to B.

In the case when Φ is of type A_2 the endotrivial group $T(B_1)$ is generated by $\Omega_{B_1}^1(\lambda)$ and the simple three dimensional G-module $L(\omega_1)$ considered as B_1-module by restriction. Since $L(\omega_1)$ is a B-module all of its syzygies $\Omega_{B_1}^n(L(\omega_1))$ stably lift to B by Proposition 5.1. ☐

When G is a reductive algebraic group scheme we can state a relationship between a G_r-module lifting stably to G and tensor stability as a direct application of [10, Theorem 1.1]. This seems to indicate that stably lifting is a stronger form of lifting that tensor stability.

Proposition 5.4. *Let G be reductive and let M be a G_r-module which lifts stably to G. Suppose that N is a G-module such that $N|_{G_r} = M \oplus P$ where P is a projective G_r-module (i.e., a stable lifting of M). If $\mathrm{soc}_{G_r} M$ is a G-submodule of N then M is tensor G_r-stable.*

In the next theorem we give a condition on the quotient H/S which insures that we can lift syzygies.

Theorem 5.5. *Let S be a finite group scheme which is a normal subgroup scheme of H. Assume that*

> (i) *If L is a simple H-module, then $L|_S$ is a simple S-module, and all simple S-modules lift to L.*

(ii) *For any simple H-module L there exists a H-module $Q(L)$ such that $Q(L)|_S$ is the projective cover $L|_S$.*

(iii) *All finite-dimensional modules for H/S are completely reducible.*

Let M be a finite-dimensional H-module. Assume that the projective cover $P(M)$ of M as an S-module lifts to an H-module and there exists a surjective H-homomorphism $P(M) \to M$. Then $\Omega_S^n(M)$ lifts to an H-module for all $n \in \mathbb{Z}$.

Proof. We begin with an observation about the cohomology. For any H-module N, there exists a Lyndon-Hochschild-Serre (LHS) spectral sequence:

$$E_2^{i,j} : \mathrm{Ext}_{H/S}^i(k, \mathrm{Ext}_S^j(k, N)) \Rightarrow \mathrm{Ext}_H^{i+j}(k, N).$$

Condition (iii) implies that this spectral sequence collapses and hence, the restriction map $\mathrm{Ext}_H^j(k, N) \to \mathrm{Ext}_S^j(k, N)^{H/S}$ is an isomorphism for all $j \geq 0$.

If suffices to assume that $n \geq 0$ and prove the theorem by induction. The case when n is negative can be handled by using a dual argument. For $n = 0$, we have that $\Omega_S^0(M) \cong M$ and for $n = 1$, $\Omega_S^1(M) \cong \ker(P(M) \to M)$. So these modules lift to H.

Suppose that $\Omega_S^n(M)$ lifts to H. The S-submodule $\mathrm{Rad}_S\Omega_S^n(M)$ is an H-submodule of $\Omega_S^n(M)$ so there exists a surjective H-module map

$$\pi : \Omega_S^n(M) \to \Omega_S^n(M)/\mathrm{Rad}_S\Omega_S^n(M).$$

The quotient module $\Omega_S^n(M)/\mathrm{Rad}_S\Omega_S^n(M)$ is completely reducible as an S-module. Furthermore, it must be completely reducible as an H-module. For if there exists a non-trivial extension of simple H-modules which lives as an H-submodule in $\Omega_S^n(M)/\mathrm{Rad}_S\Omega_S^n(M)$, then by condition (i), these simple modules remain simple upon restriction to S and this extension must split over S (by complete reducibility of the quotient module). Then by the first observation above, the original extension over H must split.

By condition (ii), there exists an H-module Q whose restriction to S is the projective cover of $\Omega_S^n(M)/\mathrm{Rad}_S\Omega_S^n(M)$ with a surjective H-module map $\gamma : Q \to \Omega_S^n(M)/\mathrm{Rad}_S\Omega_S^n(M)$. We have a short exact sequence of H-modules:

$$0 \to \mathrm{Rad}_S\Omega_S^n(M) \to \Omega_S^n(M) \to \Omega_S^n(M)/\mathrm{Rad}_S\Omega_S^n(M) \to 0.$$

Observe that $\mathrm{Ext}_H^1(Q, \mathrm{Rad}_S\Omega_S^n(M)) = \mathrm{Ext}_S^1(Q, \mathrm{Rad}_S\Omega_S^n(M))^{H/S} = 0$. Therefore, in the long exact sequence in cohomology the map

$$\mathrm{Hom}_H(Q, \Omega_S^n(M)) \to \mathrm{Hom}_H(Q, \Omega_S^n(M)/\mathrm{Rad}_S\Omega_S^n(M))$$

is surjective and we can find and H-module map $\delta : Q \to \Omega_S^n(M)$ such that $\pi \circ \delta = \gamma$ and $\Omega_S^{n+1}(M) = \ker \delta$. Consequently, $\Omega_S^{n+1}(M)$ is an H-module. $\square$

In the case that $H = G_rT$ (resp. P_rT, B_rT) and $S = G_r$ (resp. P_r, B_r) conditions (i)-(iii) of the preceding theorem can be verified (cf. [9, Chapter 9]).

Corollary 5.6. *Let $H = G_rT$ (resp. P_rT, B_rT) and $S = G_r$ (resp. P_r, B_r). If M is an H-module then for each $n \in \mathbb{Z}$, $\Omega_S^n(M)$ is an H-module.*

The conditions (i)-(iii) do not hold for when $H = G$ and $S = G_r$. Nonetheless, we can prove by a direct calculation that all endotrivial G_1-modules for $G = SL_2$ lift to G.

Theorem 5.7. *Let $G = SL_2$. Then every endotrivial G_1-module lifts to G.*

Proof. The category of G_1-modules has tame representation type and the indecomposable modules have been determined (cf. [6, Section 3]). The modules of complexity two, which include all endotrivial modules, lift to G.

One can also verify this by using the classification of endotrivial modules given in [3]. The endotrivial group is $\mathbb{Z} \oplus \mathbb{Z}_2$ and all endotrivial modules are of the form $\Omega^n(M)$ where $n \in \mathbb{Z}$ and $M = k$ or $M = L(p-2)$.

In either case $M \cong M^*$ where M^* is the k-dual of M. Hence $\Omega^n(M) \cong \Omega^{-n}(M)^*$, and without loss of generality, we may assume that $n \geq 0$. The minimal projective resolution $P_\bullet \to k$ can be constructed explicitly. All the terms are tilting modules:

$$P_n \cong \begin{cases} T((\frac{n}{2}+1)2(p-1)) & \text{if } n \text{ is even,} \\ T((\frac{n+1}{2})2p) & \text{if } n \text{ is odd.} \end{cases}$$

Moreover, the syzygies are Weyl modules

$$\Omega^n(k) \cong \begin{cases} V(np) & \text{if } n \text{ is even,} \\ V((\frac{n+1}{2})2(p-2)) & \text{if } n \text{ is odd.} \end{cases}$$

Using a similar construction, the minimal projective resolution $\widehat{P}_\bullet \to L(p-2)$ consists of tilting modules and the syzygies are also Weyl modules.

$$\widehat{P}_n \cong \begin{cases} T((n+1)p) & \text{if } n \text{ is even,} \\ T((n+2)p-2) & \text{if } n \text{ is odd,} \end{cases}$$

$$\Omega^n(L(p-2)) \cong \begin{cases} V((n+1)p-2) & \text{if } n \text{ is even,} \\ V(np) & \text{if } n \text{ is odd.} \end{cases}$$

$\square$

6. An example where $\Omega_S^2(k)$ does not lift to H

In this section we show that syzygies of the trivial module do not, in general, lift to H-modules even in cases where all the projective indecomposable S-modules lift to H. Suppose that $G = SL_3$ and that $p = 2$. Let $S = B_1$, $H = B$. The restricted p-Lie algebra $\mathfrak{u}$ of the unipotent radical of G, has the same representation theory as the infinitesimal unipotent subgroup $U_1 \subseteq B_1$. In this context we prove the following.

Proposition 6.1. *The second syzygy $\Omega^2(k) := \Omega_{B_1}^2(k)$ stably lifts to a B-module, but does not lift to a B-module.*

Proof. The first statement follows from Theorem 5.2. We suppose that $\Omega^2(k)$ has a B-structure and prove that this leads to a contradiction.

Let $V := L(\omega_1)$ be the three dimensional natural representation for G and label the simple roots $\Delta = \{\alpha_1, \alpha_2\}$. We will consider the restriction of V to B and B_1. Set

$$N_1 \cong V \otimes (-2\alpha_1 - \alpha_2 - \omega_1)$$

and

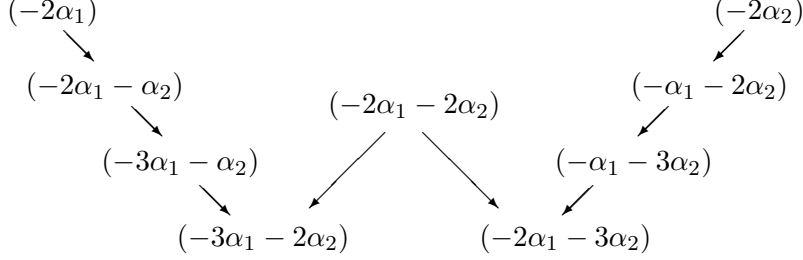$$N_2 \cong V \otimes (-\alpha_1 - 2\alpha_2 - \omega_1).$$

FIGURE 1.

We can represent $\Omega^2(k)$ diagrammatically as in Figure 1. A node (λ) is a one-dimensional B_i-submodule with highest weight λ. The arrow indicate the action of the simple root-subspace vectors in $\mathfrak{u} := \mathrm{Lie}\, U$, and the extensions between the simple one-dimensional B_1-modules. That is, an arrow that goes down and left represents multiplication by u_{α_1}, while an arrow going down and right is multiplication by u_{α_2}.

By analyzing the structure of $\Omega^2(k)$ we can conclude that

$$\Omega^2(k)/\mathrm{Rad}_{B_1}\Omega^2(k) \cong \mathfrak{u}^{(1)}$$

and

$$\mathrm{Rad}_{B_1}\Omega^2(k) \cong N_1 \oplus N_2$$

as B-modules.

The module $\Omega^2(k)$ is indecomposable over B_1. Therefore, if $\Omega^2(k)$ has a B-structure then it is indecomposable over B and represents a non-trivial extension class in

$$\mathrm{Ext}_B^1(\Omega^2(k)/\mathrm{Rad}_{B_1}\Omega^2(k), \mathrm{Rad}_{B_1}\Omega^2(k)).$$

This implies that $\mathrm{Ext}_B^1(\mathfrak{u}^{(1)}, N_j) \neq 0$ for $j = 1$ or 2.

Our task is to show by a cohomological calculation that $\mathrm{Ext}_B^1(\mathfrak{u}^{(1)}, N_j) = 0$ for $j = 1$ and 2. This provides a contradiction to the assumption that $\Omega^2(k)$ has a compatible B-structure. By symmetry we can simply look at the case that $j = 1$. Apply the LHS spectral sequence

$$E_2^{i,j} = \mathrm{Ext}_{B/B_1}^i(k, \mathrm{Ext}_{B_1}^j(\mathfrak{u}^{(1)}, N_1) \Rightarrow \mathrm{Ext}_B^{i+j}(\mathfrak{u}^{(1)}, N_1). \tag{1}$$

Note that $\mathrm{Hom}_{B_1}(k, N_1) = 0$ so the five term exact sequence (associated to this spectral sequence) yields:

$$E_1 = \mathrm{Ext}_B^1(\mathfrak{u}^{(1)}, N_1) \cong \mathrm{Hom}_{B/B_1}(\mathfrak{u}^{(1)}, \mathrm{Ext}_{B_1}^1(k, N_1)). \tag{2}$$

We can utilize the techniques in [12, Lemma 3.1.1, Theorem 3.2.1] to compute the B/B_1-socle of $\mathrm{Ext}_{B_1}^1(k, N_1)$. Observe that as a B/B_1-module:

$$\begin{aligned}
\mathrm{Ext}_{B_1}^1(k, N_1) &\cong \mathrm{Ext}_{B_1}^1(L(\omega_1)^*, -2\alpha_1 - \alpha_2 - \omega_1) \\
&\cong \mathrm{Ext}_{B_1}^1(L(\omega_2), -2\alpha_1 - \alpha_2 - \omega_1).
\end{aligned}$$

Let $-p\nu$ be a simple module in the socle where $\nu \in X$. Recall that X is the set of weights and X_+ is the set of dominant weights. Then

$$\operatorname{Hom}_{B/B_1}(-p\nu, \operatorname{Ext}^1_{B_1}(L(\lambda), \mu)) \cong \operatorname{Hom}_{B/B_1}(k, \operatorname{Ext}^1_{B_1}(L(\lambda), \mu) \otimes p\nu)$$
$$\cong \operatorname{Hom}_{B/B_1}(k, \operatorname{Ext}^1_{B_1}(L(\lambda), \mu + p\nu)).$$

Set $\lambda = \omega_2$ and $\mu = 2\alpha_1 - \alpha_2 - \omega_1$. Consider the LHS spectral sequence

$$E_2^{i,j} = \operatorname{Ext}^i_{B/B_1}(k, \operatorname{Ext}^j_{B_1}(L(\lambda), \mu + p\nu)) \Rightarrow \operatorname{Ext}^{i+j}_B(L(\lambda), \mu + p\nu).$$

So $\operatorname{Hom}_{B_1}(L(\lambda), \mu + p\nu) = 0$, because $\lambda - \mu \notin pX$. The associated five term exact sequence yields an isomorphism given as

$$E_2^{0,1} = \operatorname{Hom}_{B/B_1}(k, \operatorname{Ext}^1_{B_1}(L(\lambda), \mu + p\nu)) \cong \operatorname{Ext}^1_B(L(\lambda), \mu + p\nu).$$

There exists another spectral sequence

$$E_2^{i,j} = \operatorname{Ext}^i_G(L(\lambda), R^j\operatorname{ind}^G_B(\mu + p\nu)) \Rightarrow \operatorname{Ext}^{i+j}_B(L(\lambda), \mu + p\nu).$$

We have two cases to consider. Suppose $\mu + p\nu \in X_+$. Then by Kempf's vanishing theorem, this spectral sequence collapses and we have that

$$\operatorname{Ext}^1_B(L(\lambda), \mu + p\nu) \cong \operatorname{Ext}^1_G(L(\omega_2), H^0(\mu + p\nu)) = \operatorname{Ext}^1_G(V(\omega_2), H^0(\mu + p\nu)) = 0.$$

On the other hand, if $\mu + p\nu \notin X_+$, then the five term exact sequence yields

$$\operatorname{Ext}^1_B(L(\lambda), \mu + p\nu) \cong \operatorname{Hom}_G(L(\omega_2), R^1\operatorname{ind}^G_B(\mu + p\nu)).$$

By results of Andersen [1, Proposition 2.3], we have that $\mu + p\nu = s_\alpha \cdot \omega_2$ where $\alpha \in \Delta$, and s_α is a simple reflection. A direct computation shows that

$$\mu - s_{\alpha_1} \cdot \omega_2 = -2(\alpha_1 + \alpha_2)$$

and

$$\mu - s_{\alpha_2} \cdot \omega_2 = -3\alpha_1.$$

The second condition can not be satisfied because $-3\alpha_1 \notin pX$. Therefore, the B/B_1 socle of $\operatorname{Ext}^1_{B_1}(k, N_1)$ is one-dimensional and is equal to $-2(\alpha_1 + \alpha_2)$.

In addition, $\operatorname{Ext}^1_{B_1}(k, N_1)$ is a subquotient of $\operatorname{Hom}_{B_1}(P(k), N_1)$ (where $P(k)$ is the projective cover of k as B_1-module), and it has dimension at most two. Furthermore, the T-weights of $\mathfrak{u}^{(1)}$ are distinct and the B/B_1-socle of $\mathfrak{u}^{(1)}$ is $-2(\alpha_1 + \alpha_2)$, so the image of any any non-zero map in $\operatorname{Hom}_{B/B_1}(\mathfrak{u}^{(1)}, \operatorname{Ext}^1_{B_1}(k, N_1))$ is three-dimensional. We can now conclude that $\operatorname{Hom}_{B/B_1}(\mathfrak{u}^{(1)}, \operatorname{Ext}^1_{B_1}(k, N_1)) = 0$, and by (2), $E_1 = 0$. $\square$

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